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An analytic study of boiling heat transfer from a fin

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Abstract—Analytic expressions for the one-dimensional temperature distribution in a pin fin or a straight fin of rectangular profile are derived if various types of boiling occur simultaneously at adjacent locations on such a fin's surface. The heat transfer coefficients for the transition and nucleate boiling are taken as being the power functions of the wall superheat and that for film boiling as being constant. The number of cases analysed is 66. Some of the results obtained are compared with those of experiments carried out elsewhere. A quite reasonable degree of agreement is found between the theory and the experiment carried out in practice.

INTRODUCTION

VARIOUS types of boiling occur simultaneously at adjacent positions on the surface of a fin in many practical applications. These include low-finned boilers [1], evaporators for the regasification of cryogenic liquids [2], and cooling of large radio power tubes [3] and of semi-conductor valves [4].

Analytic studies dealing with boiling heat transfer from single fins are rarely to be found in the literature. To the author's knowledge, only five papers have so far appeared on this subject [4–8]. This is probably due to two reasons. The first being that the differential equation for temperature distribution in a fin in a boiling liquid is highly non-linear in character and consequently it appears difficult to solve by using an analytic method, if it can be solved at all [7]. The second reason is that the numerical solution of the foregoing equation is straightforward and eliminates some assumptions necessary to obtain an analytic (i.e. closed-form) solution [1, 9]. For the design engineer, however, the desirability of simple closed-form expressions may well outweigh considerations of rigor and exactness [10].

Dul'kin *et al.* [5] analytically determined one-dimensional temperature distribution in a pin (i.e. cylindrical) fin and the heat duty (i.e. the heat flux) at its base when different types of boiling occurred simultaneously on its surface. The boiling heat transfer coefficients used by these investigators are given by

$$h_j = a_j \theta_j^{N_j} \tag{1}$$

where $N_1 = 0$ for film boiling; $N_2 = -4$ for R113, $N_2 = -2.4$ for water and $N_2 = -1$ for water and R113 for transition boiling; and $N_3 = 2$ for nucleate boiling. The index j in equation (1) refers to the type of boiling. N is a non-dimensional constant and a is a dimensional constant. The results of their analytic study were verified with those of their experimental study carried out using four copper pin fins of different lengths. The boiling media used were saturated water and R113 at atmospheric pressure. Petukhov *et al.* [4] carried out work similar to that described above using $N_1 = 0$, $N_2 = -3$ and $N_3 = 2$ for R113.

The main findings reported by Dul'kin *et al.* [5] and Petukhov *et al.* [4] confirm those of Haley and Westwater [1] which were disclosed earlier. The latter measured the performances (i.e. temperature gradients and wall superheats at the fin bases) of a copper spine and two copper pin fins while boiling took place on them. The cooling media used were saturated R113 and isopropyl alcohol at atmospheric pressure. They also determined these performances with a numerical method and reported:

—that the heat rejected by a fin in a boiling liquid may be of an order of magnitude greater than for the base metal without fins; and
—that not only are three values of θ_b (the wall superheat at the fin base) possible for most heat fluxes at the fin base but also the heat flux at the fin base is a triple-valued function at certain values of θ_b, that is to say at least when the ratio of the length of the fin to its diameter is large.

If all types of saturated pool boiling are taken into account, the power in equation (1) may vary between -6.6 and 5 [11]. As is already well known, and excluding zero, a positive power in equation (1) applies to nucleate boiling and a negative power to either transition boiling or film boiling. If this power is equal to zero, then the heat transfer coefficient is constant. Ünal [7] gave the values of the power in equation (1) for which an analytic (i.e. closed-form) solution can be derived practically for one-dimensional temperature distribution in a straight fin of rectangular profile or in a pin fin. These values are: -4, -3, -2.5, -2.4, -2.2, -2.1, -1.9, -1.8, -1.6, -1.5, -1, 0, 1 and 2. With regard to the negative

NOMENCLATURE

A	cross-sectional area of a fin [m^2]
a	given constant [W m^{-2} K$^{-(N_j+1)}$]
B	constant defined in the text
b	half fin thickness [m]
C, D	integration constants
E, F, G	functions defined in the text
g	constant defined in the text
$H(\phi/\alpha)$	Legendre's normal elliptic integral of the first kind
h	heat transfer coefficient [W m^{-2} K^{-1}]
i	index (a natural number)
k	thermal conductivity of fin material [W m^{-1} K^{-1}]
L	fin length [m]
m	given dimensionless constant
N	given dimensionless constant
P	circumference of a pin fin [m]
p–s	functions defined in the text
S	first derivative of wall superheat [K m^{-1}]
t–w	constants defined in the text
x	space coordinate [m]
y	constant defined in the text
z	fin effectiveness.

Greek symbols

α	modular angle [radians]
Γ	gamma function
θ	wall superheat (i.e. difference between wall and saturation temperature) at point x [K]
ϑ, σ	real numbers
ϕ	amplitude [radians].

Subscripts

1	refers to film boiling
2	refers to transition boiling
3	refers to nucleate boiling
b	refers to fin base
d	refers to dryout location
e	refers to fin tip
f	refers to location where film boiling terminates
j	index referring to the type of boiling (j = 1, 2 and 3).

numbers, he only considered the integers and the one-digit numbers. He further derived analytic expressions for temperature distributions in the foregoing fins when the power in equation (1) equalled 1 and 2.

The object of this study is to present analytic solutions for the one-dimensional differential equation for the temperature distribution in a pin fin or in a straight fin of rectangular profile if film, transition and nucleate boiling or transition and nucleate boiling or film and transition boiling occur simultaneously at adjacent positions on the surface of the fin. To this end, the film boiling heat transfer coefficient is assumed to be constant. Excluding zero, the power in equation (1) is taken to be equal to each of the foregoing negative numbers for transition boiling, and to each of the foregoing positive numbers for nucleate boiling. The case in which only transition boiling occurs on the surface of the fin is also dealt with and thus the number of analytic solutions given for the temperature distribution in the fin is 66. The case in which only film boiling occurs on the surface of the fin (i.e. $N_1 = 0$) is given in the relevant text books. The results of the analytic work presented will be compared with those of the experimental study reported in ref. [1].

BASIC DIFFERENTIAL EQUATION OF TEMPERATURE DISTRIBUTION IN A FIN

A pin (i.e. cylindrical) fin or a straight fin of rectangular profile is now to be considered. For the analysis of such a fin, the following assumptions are made: one-dimensional, steady-state heat conduction through the fin; a constant thermal conductivity for fin material; no heat sources in the fin itself; negligible heat transfer from the fin tip; and a constant fin cross-sectional area. The liquid surrounding the fin is in a state of saturation corresponding to the system pressure. The origin of the space coordinate x is at the fin base and positive x is toward the fin tip.

For the conditions now being considered and for a particular type of boiling j, the differential equation of the temperature distribution in the fin becomes [12]

$$\frac{d^2\theta_j}{dx^2} = \frac{th_j\theta_j}{k} \tag{2}$$

where

$$t = P/A \text{ for the pin fin} \tag{3}$$

$$t = 1/b \text{ for the straight fin.} \tag{4}$$

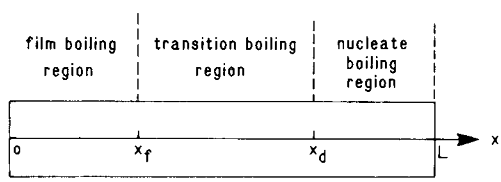
Following the procedure presented in ref. [7] and after having inserted h_j given in equation (1) into equation (2), the latter is reduced to

$$S_j\frac{dS_j}{d\theta_j} = B_j\theta_j^{(N_j+1)} \tag{5}$$

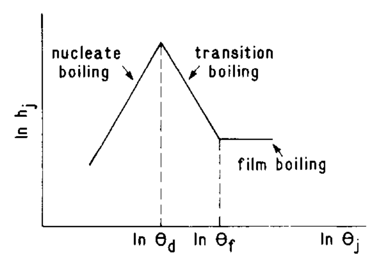
where

$$S_j = \frac{d\theta_j}{dx} \tag{6}$$

$$B_j = \frac{a_j t}{k}. \tag{7}$$



a) Simultaneous occurrence of the three types of boiling on a fin



b) Boiling curve used

FIG. 1. Characteristics of a fin in a boiling liquid.

In order to determine the temperature distribution in the fin whilst the various types of boiling occur together at adjacent positions on its surface, a set of differential equations should be solved simultaneously as will be discussed in the following sections.

SOLUTION OF THE DIFFERENTIAL EQUATIONS

Film, transition and nucleate boiling

If these three types of boiling occur together on the surface of the fin as illustrated in Fig. 1(a), then three differential equations should be simultaneously solved in order to determine the temperature distribution in it. These equations are obtained with equation (5) if the index j in the latter is taken as being equal to 1, 2 and 3 for film, transition and nucleate boiling, respectively. They are given by

$$S_1 \frac{dS_1}{d\theta_1} = B_1\theta_1 \tag{8}$$

$$S_2 \frac{dS_2}{d\theta_2} = B_2\theta_2^{(N_2+1)} \tag{9}$$

$$S_3 \frac{dS_3}{d\theta_3} = B_3\theta_3^{(N_3+1)}. \tag{10}$$

The boundary conditions for equation (8) are expressed in equations (11) and (12), those for equation (9) in equations (13) and (14) and those for equation (10) in equations (15) and (16). The boiling curve used is shown in Fig. 1(b).

$$\theta_1 = \theta_b \qquad \text{for } x = 0 \tag{11}$$

$$-\frac{d\theta_1}{dx} = -\frac{d\theta_2}{dx} \qquad \text{for } x = x_f \tag{12}$$

$$\theta_2 = \theta_f \qquad \text{for } x = x_f \tag{13}$$

$$-\frac{d\theta_2}{dx} = -\frac{d\theta_3}{dx} \qquad \text{for } x = x_d \tag{14}$$

$$\theta_3 = \theta_d \qquad \text{for } x = x_d \tag{15}$$

$$-\frac{d\theta_3}{dx} = 0 \qquad \text{for } x = L. \tag{16}$$

θ_f, the wall superheat at the location where film boiling terminates, θ_d, the wall superheat at the dryout (i.e. burnout) location, and $\mathbf{a}_j$ and N_j in equation (1) (i.e. a_1, a_2, a_3, N_1, N_2 and N_3) are known. x_f, x_d, $x = x(\theta_1)$ for $0 \leqq x \leqq x_f$; $x = x(\theta_2)$ for $x_f \leqq x \leqq x_d$; and $x = x(\theta_3)$ for $x_d \leqq x \leqq L$, will be determined.

After rearrangement, and taking into consideration equation (6) and the fact that the temperature in the fin decreases along its length, the first integration of equations (8)–(10) yield the temperature gradient in film, transition and nucleate boiling region, respectively.

$$\frac{d\theta_1}{dx} = -(B_1\theta_1^2 + C_1)^{0.5} \tag{17}$$

$$\frac{d\theta_2}{dx} = -\left(\frac{2B_2\theta_2^{(N_2+2)}}{N_2+2} + C_2\right)^{0.5} \qquad \text{for } N_2 \neq -2 \tag{18}$$

$$\frac{d\theta_3}{dx} = -\left(\frac{2B_3\theta_3^{(N_3+2)}}{N_3+2} + C_3\right)^{0.5} \tag{19}$$

if each of the integration constants obtained is replaced by a new integration constant.

The constant of integration C_1 in equation (17) is calculated using boundary condition given in equation (12), and equations (17) and (18) as

$$C_1 = C_2 + \frac{2B_2\theta_f^{(N_2+2)}}{N_2+2} - B_1\theta_f^2 \tag{20}$$

since $\theta_1 = \theta_2 = \theta_f$ for $x = x_f$. C_2 and C_3, the integration constants in equations (18) and (19) respectively, are obtained in a manner analogous to that described above, using the boundary conditions expressed in equations (14) and (16), respectively. These are given by

$$C_2 = C_3 + \frac{2B_3\theta_d^{(N_3+2)}}{N_3+2} - \frac{2B_2\theta_d^{(N_2+2)}}{N_2+2} \tag{21}$$

$$C_3 = -\frac{2B_3\theta_e^{(N_3+2)}}{N_3+2}. \tag{22}$$

The only unknown value in equations (20)–(22) is θ_e (the wall superheat at the fin tip). The practical significance of these equations is obvious: If θ_e is known, then the temperature gradient (or the heat flux) at the fin base can be determined using equation (17) for all values of N_j, excluding $N_2 = -2$. The foregoing implies that the one-dimensional numerical analysis of the fin being considered may be substantially shortened.

In order to calculate the temperature distribution in the film boiling region, equation (17) is integrated to give

$$\frac{1}{\sqrt{B_1}}\ln[D_1(\theta_1\sqrt{B_1} + \sqrt{B_1\theta_1^2 + C_1})] = -x. \quad (23)$$

The constant of integration in equation (23), D_1, is determined using boundary condition given in equation (11):

$$D_1 = (\theta_b\sqrt{B_1} + \sqrt{B_1\theta_b^2 + C_1})^{-1} \quad (24)$$

x_f, the location where film boiling terminates is obtained with equation (23) as

$$x_f = -\frac{1}{\sqrt{B_1}}\ln[D_1(\theta_f\sqrt{B_1} + \sqrt{B_1\theta_f^2 + C_1})] \quad (25)$$

since $\theta_1 = \theta_f$ for $x = x_f$.

It follows from equations (23)–(25) that θ_1 for a given x (or x for a given θ_1) for $0 \leqq x \leqq x_f$ and x_f can be predicted if θ_e is known.

In order to find the temperature distribution in the transition boiling region, equation (18) should be integrated. The integration of this equation can be analytically made for a few values of N_2, which were already mentioned. For these values of N_2, this integration can be straightforwardly carried out using a mathematical handbook [13]. The result of the integration is given by

$$E(\theta_2) = -x + D_2 \quad (26a)$$

for $N_2 = -1, -1.5, -1.6, -1.8, -1.9, -2.4$ and -4, and by

$$F(\theta_2) + g\ln[D_2G(\theta_2)] = -x \quad (26b)$$

for $N_2 = -2.1, -2.2, -2.5$ and -3. E in equation (26a), and F and G in equation (26b) are functions of θ_2. g in the latter equation is a constant. D_2 in both equations is the constant of integration. E, F, G and g are given in the Appendix for the values of N_2 given above.

For the transition boiling region, the a-version of an equation applies to the values of N_2 mentioned first and the b-version to the values of N_2 mentioned later.

Using the boundary condition expressed in equation (13), the constant of integration in equation (26a) is determined as

$$D_2 = x_f + E(\theta_f) \quad (27a)$$

and that in equation (26b) as

$$D_2 = \frac{1}{G(\theta_f)}\exp\left(\frac{-x_f - F(\theta_f)}{g}\right). \quad (27b)$$

x_d, the location of dryout, is determined with equation (26a) as

$$x_d = D_2 - E(\theta_d) \quad (28a)$$

and with equation (26b) as

$$x_d = -F(\theta_d) - g\ln[D_2G(\theta_d)] \quad (28b)$$

since $\theta_2 = \theta_d$ for $x = x_d$.

It follows from equations (26)–(28) that θ_2 for a given x (or x for a given θ_2) for $x_f \leqq x \leqq x_d$ and x_d can be predicted if θ_e is known.

The temperature distribution in the nucleate boiling region is obtained by integrating equation (19). As noted earlier herein, the integration of this equation is impracticable with analytic methods except in the cases where $N_3 = 1$ and 2. Introducing a new variable $p = \theta_3/\theta_e$, equation (19) is reduced to

$$\frac{dp}{(p^{(N_3+2)} - 1)^{0.5}} = -y\,dx \quad (29)$$

where

$$y = \left(\frac{2B_3\theta_e^{N_3}}{N_3 + 2}\right)^{0.5}. \quad (30)$$

The integration of equation (29) can be made using a mathematical handbook [13, 14]. Neglecting the details, the integration of the equation is given below:

$$mH(\phi/\alpha) = -yx + D_3 \quad (31)$$

where

$$m = 2^{-0.5} \quad \text{for } N_3 = 2 \quad (32a)$$

$$\alpha = \pi/4 \quad \text{for } N_3 = 2 \quad (33a)$$

$$\phi = \arccos(\theta_e/\theta_3) \quad \text{for } N_3 = 2 \quad (34a)$$

$$m = 3^{-0.25} \quad \text{for } N_3 = 1 \quad (32b)$$

$$\alpha = \pi/12 \quad \text{for } N_3 = 1 \quad (33b)$$

$$\phi = \arccos\left(\frac{\sqrt{3} + 1 - \theta_3/\theta_e}{\sqrt{3} - 1 + \theta_3/\theta_e}\right) \quad \text{for } N_3 = 1. \quad (34b)$$

$H(\phi/\alpha)$ in equation (31), Legendre's (incomplete) normal elliptic integral of the first kind, is tabulated in ref. [14] as a function of ϕ and α. This integral is also given as an analytic function in refs. [7, 13]. $H(\phi/\alpha)$ is valid $0 \leqq \phi \leqq \pi$. For the nucleate boiling region, the a-version of an equation refers to $N_3 = 2$ and the b-version to $N_3 = 1$.

Using the boundary condition expressed in equation (15), the constant of integration in equation (31) is determined as

$$D_3 = mH(\phi_d/\alpha) + yx_d \quad (35)$$

where

$$\phi_d = \arccos(\theta_e/\theta_d) \quad \text{for } N_3 = 2 \quad (36a)$$

$$\phi_d = \arccos\left(\frac{\sqrt{3} + 1 - \theta_d/\theta_e}{\sqrt{3} - 1 + \theta_d/\theta_e}\right) \text{for } N_3 = 1. \quad (36b)$$

The temperature distribution in the nucleate boiling region is expressed in equation (31) and can be determined if θ_e is known.

It thus follows that the temperature distributions in the film, transition and nucleate boiling regions can be calculated if θ_e is known. In order to predict these temperature distributions, x_f and x_d, the following procedure is adopted: for $x = L$, equation (31) is reduced to

$$yL = mH(\phi_d/\alpha) + yx_d \tag{37}$$

since $H(\phi_e/\alpha) = 0$. A value for θ_e is first assumed, thereafter x_f is determined with equation (25) and x_d with equation (28). θ_e is iterated until equation (37) is satisfied. Having determined θ_e, all the constants of integrations are then known; thus the temperature distribution in the film, transition and nucleate boiling region can be determined with equations (23), (26) and (31), respectively. These equations are not explicit but implicit functions of the independent variable x. Therefore it is a straightforward matter to calculate x for a given θ_j since $\theta_b \geqq \theta_1 \geqq \theta_f$, $\theta_f \geqq \theta_2 \geqq \theta_d$ and $\theta_d \geqq \theta_3 \geqq \theta_e$.

The temperature gradient at the fin base, one of the most significant of the criteria to characterize the performance of the fin, is predicted with equation (17), with the value $\theta_1 = \theta_b$. The heat flux at the fin base is obtained by multiplying the reverse of this temperature gradient with the thermal conductivity of fin material.

Nucleate and transition boiling

If these two types of boiling occur together at adjacent positions on the surface of the fin, equations (9) and (10) should be solved simultaneously. The boundary conditions for equation (10) [i.e. equations (15) and (16)] and the boundary condition expressed in equation (14) for equation (9) hold good. The other boundary condition for equation (9) is given by

$$\theta_2 = \theta_b \qquad \text{for } x = 0. \tag{38}$$

The simultaneous solution of equations (9) and (10) is analogous to that of equations (8)–(10). Omitting the details, the results of the solution are presented below.

The temperature distribution in the transition boiling region is again expressed in equation (26). C_2 and C_3 given by equations (21) and (22) hold good. D_2 given by equations (27a) and (27b) should be replaced by equations (39a) and (39b), respectively.

$$D_2 = E(\theta_b) \tag{39a}$$

$$D_2 = \frac{1}{G(\theta_b)}\exp[-F(\theta_b)/g]. \tag{39b}$$

Equations (28a) and (28b) also hold good.

The temperature distribution in the nucleate boiling region is again given by equation (31), equations (32)–(37) being valid here.

In order to predict the temperature distributions in the transition and nucleate boiling regions and the dryout location x_d, the following procedure is adopted. A value for θ_e is assumed. x_d is determined with equation (28). θ_e is iterated until equation (37) is satisfied. Having calculated θ_e, the temperature distribution in the transition boiling region is then evaluated with equation (26) and that in the nucleate boiling region with equation (31). The temperature gradient at the fin base is given by equation (18) if θ_2 in it is replaced by θ_b.

Film and transition boiling

If these two types of boiling occur together at adjacent locations on the surface of the fin, the temperature distribution in it can be determined by solving equations (8) and (9) simultaneously. The boundary conditions expressed in equations (11) and (12) for equation (8) and the boundary condition expressed in equation (13) for equation (9) hold good. The other boundary condition for the latter equation is given by

$$\frac{d\theta_2}{dx} = 0 \qquad \text{for } x = L. \tag{40}$$

Neglecting the details in the simultaneous solution of equations (8) and (9), the results of the solution are outlined below.

C_2 is determined with equation (18) using the boundary condition expressed in equation (40):

$$C_2 = -\frac{2B_2\theta_e^{(N_2+2)}}{N_2+2} \tag{41}$$

C_1, given by equation (20), holds good if C_2 in it is calculated with equation (41). The temperature distribution in the film boiling region is evaluated with equation (23), and equations (24) and (25) are valid in this case. The temperature distribution in the transition boiling region is again predicted with equation (26), equation (27) holding good here.

In order to determine the temperature distribution in the fin and x_f, the following method is used. Noting that x equals the fin length L for $\theta_2 = \theta_e$, a value for θ_e is assumed. x_f is predicted with equation (25) and x with equation (26). θ_e is iterated until the calculated x is equal to L. Having found the value of θ_e, the temperature distribution in the film boiling is then determined with equation (23) and that in the transition boiling region with equation (26). The temperature gradient at the fin base is evaluated with equation (17), taking $\theta_1 = \theta_b$ in it.

Transition boiling

If only this type of boiling occurs on the surface of the fin, the temperature distribution in it is obtained by integrating equation (9). The boundary conditions are given in equations (38) and (40). The first integration of equation (9) results in equation (18). C_2 in the latter equation is determined using the boundary condition expressed in equation (40). This value of C_2 is given by equation (41). The integration of equation

Table 1. Boiling data used

a_j, N_j, θ_f and θ_d	Isopropyl alcohol	R113	Type of boiling
a_1, W m^{-2} K^{-1}	254	194	film
a_2, W m^{-2} K$^{-(N_2+1)}$	4.7×10^7	3×10^9	transition
a_3, W m^{-2} K^{-3}	28	16	nucleate
N_1	0	0	film
N_2	-2.5	-4	transition
N_3	2	2	nucleate
θ_f, K	81	71	--
θ_d, K	22.7	22	—

(18) yields the temperature distribution in the fin which is again given by equation (26). Using the remaining boundary condition, the constant of integration in equation (26) is evaluated as expressed in equation (39). In order to utilize equation (26), θ_e in it should be determined. Noting that x equals the fin length L for $\theta_2 = \theta_e$, a value for θ_e is assumed, and x is predicted with equation (26). θ_e is iterated until the calculated value of x is equal to L. The temperature gradient at the fin base is determined with equation (18), taking the value $\theta_2 = \theta_b$.

VERIFICATION WITH AVAILABLE DATA

For this purpose, the previously quoted data produced by Haley and Westwater [1] with 19.6- and 30.7-mm-long, horizontal copper pin fins of 6.35 mm o.d. were used. These investigators measured the temperature gradients and wall superheats at the bases of these fins whilst various types of saturated pool boiling occurred at adjacent locations on their surfaces at atmospheric pressure. Tests were made with the 30.7-mm-long fin in R113 and in isopropyl alcohol. Additional tests were made with the 19.6-mm-long fin in R113.

In order to calculate the temperature distribution in a fin in a boiling liquid, a_j and N_j in equation (1), and θ_f and θ_d should be known. Although the surface of such a fin is non-isothermal, Haley and Westwater [1], and Petukhov *et al.* [4] evaluated a_j, N_j, θ_f and θ_d from the data obtained on isothermal surfaces. The first quoted of these investigator teams concluded that this is a good procedure, while Dul'kin *et al.* [5] reported that N_2 for transition boiling on a non-isothermal surface is much higher than that on an isothermal surface (i.e. -1 to -4 for R113 and -1 to -2.4 for water).

In the present study, a_j, N_j, θ_f and θ_e were derived from the data measured on isothermal surfaces. The data quoted in ref. [15] from 11 different experimental studies were considered for isopropyl alcohol and the data of [1] for R113. The values of a_j and N_j were obtained with a curve fitting technique. These values and those of θ_d and θ_f are given in Table 1. The thermal conductivity of copper was taken as being equal to 382 W m^{-1} K^{-1}.

The calculated and measured temperature gradients at the bases of the 19.6- and 30.7-mm-long fins in boiling R113 are shown plotted against θ_b in Fig. 2. The calculations were carried out with a programmable desk calculator with 224 program steps. Consider now the 30.7-mm-long fin and assume that θ_b is slowly increased. If $\theta_d < \theta_b \leq \theta_f$, transition and nucleate boiling occur simultaneously on the fin, i.e. the BC part of the curve shown in the figure. The AB part of the curve corresponds to the case in which only nucleate boiling occurs on the fin. If $\theta_f < \theta_b(\mathrm{K}) \leq 101.3$, film, transition and nucleate boiling take place simultaneously on the fin, i.e. the CD part of the curve. If θ_b is increased beyond 101.3 K, film boiling occurs on the fin alone, i.e. the FG part of the curve. Film boiling also occurs on the fin only if $82 \leq \theta_b(\mathrm{K}) \leq 101.3$, i.e. the EF part of the curve. The dashed-line part of the curve (i.e. the DE part) corresponds to the cases in which film and transition boiling occur simultaneously on the fin and that transition boiling exists on the fin only. Haley and Westwater [1] reported that it was not possible to operate along this dashed-line part experimentally. The calculated temperature gradients given in Fig. 2 fit the data slightly better than the temperature gradients calculated with a numerical method in ref. [1].

In Fig. 3, the calculated and measured temperature gradients at the base of the 30.7-mm-long fin in boiling isopropyl alcohol are shown plotted against θ_b. The curve given in the figure is analogous to that shown in Fig. 2 for the 30.7-mm-long fin, with the exception that for no θ_b, transition boiling occurs on the fin only. In Fig. 3, the temperature gradient curve obtained with a numerical method for the same fin [1] is also given. The two curves in this figure are practically identical. HI-portions of the curves coincide.

It follows from Figs. 2 and 3, therefore, that the analytic method presented herein appears to be quite good for the cases in which transition and nucleate boiling or film, transition and nucleate boiling occur simultaneously on a fin and that only nucleate boiling occurs on the fin. For the case in which film boiling occurs only on the fin, the method seems to be quite a reasonable one to use.

The fin effectiveness is a proper criterion to use in the evaluation of the heat-transfer performance of a surface with and without a fin. This is defined by

$$z = \frac{-k(\mathrm{d}\theta_j/\mathrm{d}x)_{x=0}}{h_j(\theta_b)\cdot\theta_b} \tag{42}$$

in which the numerator gives the heat flux at the fin base and the denominator gives the heat flux at the surface in the absence of the fin. The fin effectiveness for the 30.7-mm-long fin was calculated when the three types of boiling occurred on its surface. This value varies between 56.4 and 112.2 in the case of boiling isopropyl alcohol and between 64.2 and 95.8 in the case of boiling R113. The foregoing implies

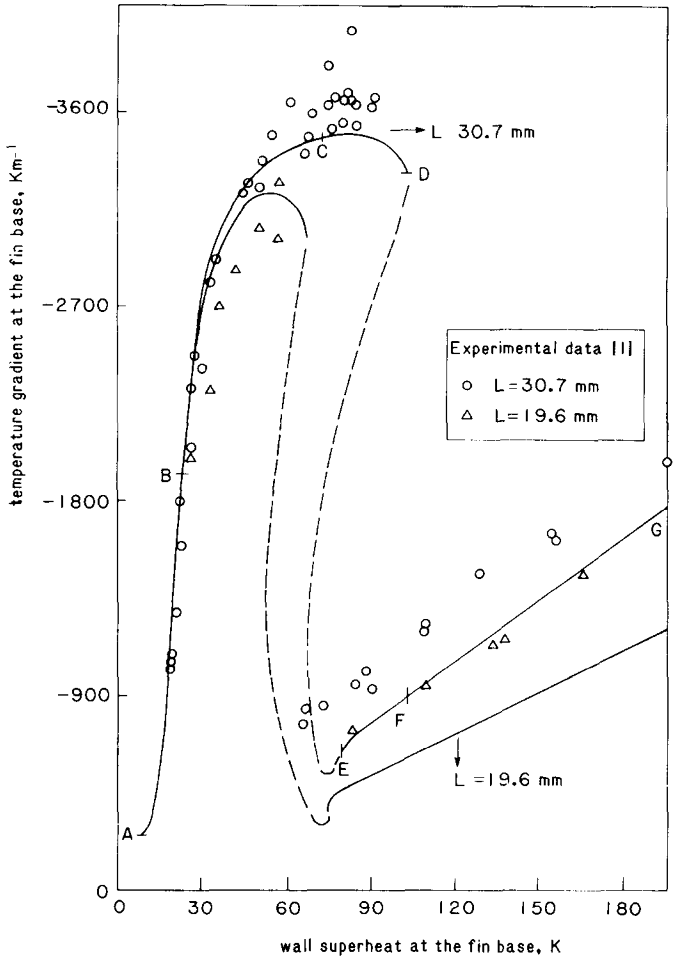


FIG. 2. Calculated and measured temperature gradients in two fins in boiling R113.

that a fin in a boiling liquid is a very effective heat transfer enhancement device for the applications in which film boiling occurs on the bare surface.

SUMMARY AND CONCLUDING REMARKS

Analytic expressions for one-dimensional temperature distribution in a pin fin or in a straight fin of rectangular profile are derived if film, transition and nucleate boiling or film and transition boiling or transition and nucleate boiling, occur simultaneously at adjacent locations on the surface of the fin. The condition in which transition boiling occurs only on the surface of the fin is also dealt with. The heat transfer coefficients in transition and nucleate boiling regions are taken as being the power functions of the wall superheat and that in the film boiling region as being constant. The number of cases amounts to 66. The results of some of these cases are compared with those of an experimental study carried out elsewhere. A quite reasonable degree of agreement was found between the theory and the experimental results found in practice.

Throughout this study, the fin tip was assumed to be insulated and that the thermal conductivity of the fin material was constant. These are not restrictive assumptions however. It is a straightforward matter to modify the expressions presented herein if the physically true boundary condition for the fin tip is considered and if the thermal conductivity of the fin material is constant but different for each of the adjacent boiling regions on the fin.

A fin in a boiling liquid appears to be a very effective heat transfer enhancement device for the applications in which film boiling occurs.

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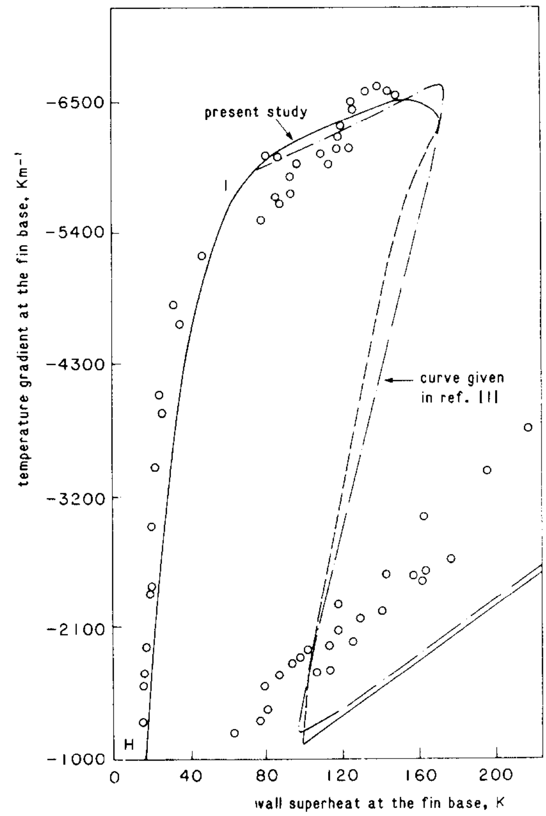

FIG. 3. Calculated and measured temperature gradients in the 30.7-mm-long fin in boiling isopropyl alcohol.

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APPENDIX

Integration of equation (18)

In order to integrate this equation, a new variable r is introduced and it is then reduced to

$$\frac{r^u \mathrm{d}r}{w(vr + C_2)^{0.5}} = -\mathrm{d}x \qquad \text{(A1)}$$

where

$$w = N_2 + 2 \qquad \text{(A2)}$$

$$r = \theta_2^w \qquad \text{(A3)}$$

$$u = -(N_2 + 1)/(N_2 + 2) \qquad \text{(A4)}$$

$$v = 2B_2/(N_2 + 2). \qquad \text{(A5)}$$

The integration of equation (A1) can be carried out using a mathematical handbook [13] and is given either by equation (26a) or by equation (26b). E, F, G and g in these equations are presented below.

Let q and s be defined as follows:

$$q = (v\theta_2^{(N_2+2)} + C_2)^{0.5} \quad \text{(A6)}$$

$$s = q\theta_2^{0.5(N_2+2)} \quad \text{(A7)}$$

For $N_2 = -1$, -1.5, -1.8 and -1.9, $E(\theta_2)$ in equation (26a) is given by

$$E(\theta_2) = \frac{1}{w}\frac{2(-C_2)^u}{v^{(u+1)}}q\sum_{i=0}^{u}\frac{(-1)^i}{2i+1}\binom{u}{i}\left(\frac{q^2}{C_2}\right)^i \quad \text{(A8)}$$

where

$$\binom{u}{i} = \frac{u!}{i!(u-i)!}. \quad \text{(A9)}$$

For $N_2 = -1.6$, $N_2 = -2.4$ and $N_2 = -4$, $E(\theta_2)$ is given by the following equations, respectively

$$E(\theta_2) = \frac{1}{w}\left\{\left(\frac{\theta_2^w}{2v} - \frac{3C_2}{4v^2}\right)s + \frac{3C_2^2}{8v^{2.5}}\ln\left(\frac{v\theta_2^w + (C_2/2)}{v^{0.5}} + s\right)\right\} \quad \text{(A10)}$$

$$E(\theta_2) = \frac{s}{w\theta_2^w}\left(-\frac{2}{5C_2\theta_2^{2w}} + \frac{8v}{15C_2^2\theta_2^w} - \frac{16v^2}{15C_2^3}\right) \quad \text{(A11)}$$

$$E(\theta_2) = -\frac{2s}{wC_2\theta_2^w}. \quad \text{(A12)}$$

For $N_2 = -2.1$, -2.2, -2.5 and -3, $F(\theta_2)$, $G(\theta_2)$ and g in equation (26b) are given by

$$F(\theta_2) = \frac{q}{wv}\sum_{i=0}^{(-u-2)}\frac{(-2u-3,-2,i)}{2^i(-u-1,-1,i+1)}\left(-\frac{v}{C_2}\right)^{(i+1)}\theta_2^{w(u+i+1)} \quad \text{(A13)}$$

$$G(\theta_2) = \frac{q - C_2^{0.5}}{q + C_2^{0.5}} \quad \text{(A14)}$$

$$g = \frac{(1,2,-u-1)2^{(u+1)}}{w(-u-1)!C_2^{0.5}}\left(-\frac{v}{C_2}\right)^{(-u-1)}. \quad \text{(A15)}$$

The special mathematical symbol used in $F(\theta_2)$ and g is defined as follows: Let ϑ and σ be real numbers and i a natural number. Then

$$(\vartheta,\sigma,i) = \vartheta(\vartheta+\sigma)(\vartheta+2\sigma)\ldots(\vartheta+(i-1)\sigma). \quad \text{(A16)}$$

For $i = 0$, $(\vartheta,\sigma,0) = 1$. The following relation also holds:

$$(\vartheta,-\sigma,i) = \sigma^i\left[\Gamma\left(\frac{\vartheta}{\sigma}+1\right)\right]\left[\Gamma\left(\frac{\vartheta}{\sigma}-i+1\right)\right]. \quad \text{(A17)}$$

UNE ETUDE ANALYTIQUE DU TRANSFERT DE CHALEUR PAR EBULLITION A PARTIR D'UNE AILETTE

Résumé—Des formules analytiques pour la distribution unidirectionnelle de température dans une aiguille ou une ailette droite à profil rectangulaire sont obtenues quand différents types d'ébullition apparaissent simultanément sur la surface. Les coefficients de transfert de chaleur pour la transition et l'ébullition nucléée sont sous forme de fonctions puissances de la surchauffe de la paroi et celui pour l'ébullition en film est constant. On analyse 66 cas. Quelques résultats obtenus sont comparés avec ceux d'expériences connues. On trouve un degré raisonnable de compatibilité entre la théorie et les expériences.

ANALYTISCHE UNTERSUCHUNG ZUM WÄRMEÜBERGANG BEIM SIEDEN AN RIPPEN

Zusammenfassung—Analytische Beziehungen für die eindimensionale Temperaturverteilung in einer Stabrippe oder einer geraden Rechteckrippe wurden für den Fall hergeleitet, daß verschiedene Verdampfungsarten gleichzeitig an unterschiedlichen Stellen der Rippenoberfläche auftreten. Die Wärmeübergangskoeffizienten für das Sieden im Übergangsbereich und für das Blasensieden wurden als Potenzfunktionen der Wandüberhitzung angenommen, der Wärmeübergangskoeffizient für das Filmsieden als konstant angesetzt. 66 Fälle wurden untersucht. Einige der Ergebnisse wurden mit experimentellen Werten anderer Autoren verglichen. Es ergab sich eine recht gute Übereinstimmung zwischen Theorie und Experiment.

АНАЛИТИЧЕСКОЕ ИССЛЕДОВАНИЕ ТЕПЛООТДАЧИ РЕБРА ПРИ КИПЕНИИ

Аннотация—Получены аналитические выражения, описывающие одномерное распределение температур в игольчатом или плоском ребре прямоугольного профиля в случае, когда одновременно имеют место различные режимы кипения на смежных участках поверхностей таких ребер. Предполагается, что коэффициенты теплопереноса для переходного и пузырькового режимов кипения являются степенными функциями перегрева стенки, но остаются постоянными при пленочном кипении. Проведен анализ 66 случаев. Дано сравнение некоторых из полученных результатов с экспериментальными данными других авторов. Показано, что наблюдается удовлетворительное совпадение теории с экспериментом.